Topological quantization of gravitational fields

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Abstract

We introduce the method of topological quantization for gravitational fields in a systematic manner. First we show that any vacuum solution of Einstein's equations can be represented in a principal fiber bundle with a connection that takes values in the Lie algebra of the Lorentz group. This result is generalized to include the case of gauge matter fields in multiple principal fiber bundles. We present several examples of gravitational configurations that include a gravitomagnetic monopole in linearized gravity, the C-energy of cylindrically symmetric fields, the Reissner-Nordström and the Kerr-Newman black holes. As a result of the application of the topological quantization procedure, in all the analyzed examples we obtain conditions implying that the parameters entering the metric in each case satisfy certain discretization relationships.

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I. INTRODUCTION

Dirac's original idea[1] of determining the phase acquired by a charge when moving along a closed path in the field of a magnetic monopole, gave as a result that the product of the electric charge times the monopole charge is an integer. Today, this result is known as Dirac's quantization of the electric charge. In a previous work[2], we introduced a phase-like object which depends on the field strength so that it can be used to investigate any field theory based on a connection. In the special case of the Levi-Civita connection, the field strength is given by the Riemann tensor and the phase-like object can formally be used to investigate the properties of gravitational configurations that satisfy Einstein's equations. Using only the symmetry properties of the Riemann tensor and assuming a quite general symmetry property for the gravitational field, we have shown that this phase-like object for vacuum gravitational configurations behave under rotations either as a bosonic or a fermionic phase. It was also shown that a certain combination of the eigenvalues of the Riemann tensor can become "quantized" in a fashion similar to that obtained from Dirac's quantization procedure in the system composed of an electric charge and a magnetic monopole.

From the geometric point of view[3], Dirac's quantization is interpreted as a consequence of the existence of a non trivial principal fiber bundle of $U(1)$ over the sphere S^2, with a $u(1)-$connection, for the system composed of an electric charge q and a magnetic monopole with magnetic charge g. The Chern numbers associated with this non trivial fiber bundle turn out to be given as the product qg which, therefore, becomes quantized.

In this work, we introduce the method of topological quantization which can be applied to any field configuration whose geometrical structure allows the existence of a principal fiber bundle. We show that any solution of Einstein's equations minimally coupled to any gauge matter field can be represented geometrically as a principal fiber bundle with spacetime as the base space. The structure group (isomorphic to the standard fiber) follows from the invariance of the orthonormal frame with respect to Lorentz transformations, in the case of a vacuum solution, or with respect to a transformation of the gauge group, in the case of a gauge matter field. If the bundle turns out to be (globally) non trivial, the conditions under which this construction becomes well-defined in all the points where the field configuration exists, manifest themselves in the transition functions between different but intersecting open subsets of the base manifold of the bundle. These conditions on the transition func-

tions turn out to depend on the parameters which determine the physical structure of the field configuration. Consequently, the conditions that arise in the construction of a fiber bundle lead to conditions on the physical parameters which, in turn, implies that a particular combination of those parameters can take only *discrete* values. This discretization can be derived also from the topological invariants of the corresponding non trivial fiber bundle. This is what we call the *quantization conditions* for a given field configuration. Furthermore, we will see that even in the case of a globally trivial principal fiber bundle certain quantization conditions may appear as a result of demanding regularity of the connection.

In Section II, we introduce the method of topological quantization in a systematic manner and briefly discuss the general cases in which non trivial quantization conditions may appear. In Section III we analyze the $so(1,3)$-connection of cylindrically symmetric gravitational fields and show that the corresponding C-energy can take only discrete values. In Section IV we present the example of a gravitomagnetic monopole in linearized Einstein's theory which can be investigated by means of a $u(1)$-connection. Section V contains the topological quantization with respect to the electromagnetic connection of gravitational fields which represent electrovacuum black holes. Section VI is devoted to discussions and remarks about future investigations.

II. THE METHOD OF TOPOLOGICAL QUANTIZATION

Consider a Riemannian manifold (M, g), where M is a 4-dimensional differential manifold and g is a bilinear form, the metric, on M. For the purpose of analyzing field equations we choose in M a set local orthonormal 1-forms e^a, $a = 0, 1, 2, 3$. The orthonormality condition can be expressed in terms of the local Minkowski metric as $g(e^a, e^b) = \eta^{ab}$, where $\eta^{ab} = \text{diag}(+, -, -, -)$. On a torsion free manifold we can introduce a connection 1-form ω by means of the Cartan first structure equation

$$De^a := de^a + \omega^a{}_b \wedge e^b = 0 \ , \tag{1}$$

where d is the exterior derivative and D the covariant exterior derivative. We demand that ω be a metric connection, i.e. locally $D\eta_{ab} = d\eta_{ab} + \omega_{ab} + \omega_{ba} = 0$, a condition which implies the antisymmetry of the connection components. Furthermore, we use the Cartan second

structure equation to introduce the curvature 2-form Ω as

$$D\omega^a{}_b := \Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \; , \tag{2}$$

whose components in terms of the local orthonormal frame, $\Omega^a{}_b = (1/2)R^a{}_{bcd}e^c \wedge e^d$, determine the Riemann curvature tensor $R^a{}_{bcd}$. Einstein's gravity theory follows from the variation, with respect to the orthonormal frame e^a of the action

$$S = -\frac{1}{32\pi G}\int_M \Omega^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} + \int_M \mathcal{L}_m \; , \tag{3}$$

where $\epsilon_{0123} = 1$ and $\mathcal{L}_m$ is the matter Lagrangian which depends on e, ω, and the matter fields. The field equations are $\Omega^{ab} \wedge e^c \epsilon_{abcd} = -16\pi G T_c$, where T_c is the energy-momentum 3-form which follows from the variation of the matter action.

The advantage of using a local orthonormal frame e is that the gauge character (at the level of the connection and curvature) of Einstein's theory becomes more plausible (see, for instance,[3,4] for a more detailed discussion). Indeed, the diffeomorphism invariance of the theory is now reduced to the invariance with respect to the Lorentz group $SO(1,3)$. The change to a different frame $e' = \Lambda e$ is represented by means of a matrix $\Lambda \in SO(1,3)$. The connection 1-form and the curvature 2-form take values in the corresponding Lorentz algebra $so(1,3)$, and under a change of frame they transform as

$$\omega' = \Lambda\omega\Lambda^{-1} + \Lambda d\Lambda^{-1} \; , \qquad \Omega' = \Lambda\Omega\Lambda^{-1} \; , \tag{4}$$

respectively. It is in this sense that Einstein's theory can be considered as a gauge theory with respect to the Lorentz group. However, it is at the level of the action that Einstein's theory tremendously differs from pure Yang-Mills gauge theories.

Let us now consider the matter action. As we have mentioned before the matter Lagrangian can depend on the frame e, the connection ω, and the matter fields. We will assume that the matter fields are gauge fields, that is, there exists a connection 1-form A, with values in the Lie algebra of a Lie group G, which generates the gauge field strength F in the standard manner: $F = dA + A \wedge A$. Under a gauge transformation $\gamma \in G$, these quantities behave as (we assume that the gauge group is a matrix group):

$$A \rightarrow A' = \gamma A\gamma^{-1} + \gamma d\gamma^{-1} \; , \qquad F \rightarrow F' = \gamma F\gamma^{-1} \; . \tag{5}$$

A vacuum spacetime in general relativity is a solution of Einstein's vacuum equations, represented by an orthonormal frame e. Since we assume that the compatibility condition

between the local metric and the connection is satisfied, we can use the connection ω, instead of the orthonormal frame. Moreover, if we adopt the Palatini approach, the connection ω can be considered as the "primary" variable, whereas the orthonormal frame e can be derived from the metricity condition. In the presence of a matter field, one needs additionally the "matter" connection A which satisfies Einstein's equations, with the corresponding energy-momentum 3-form, and the matter field equations that follow from the variation of the matter action with respect to the connection A. For a particular spacetime to be well defined we have to guarantee that ω and A are well defined everywhere in M. This is a non trivial remark for the analysis of gravitational fields we want to perform. Indeed, the fact that the connection is demanded to be well defined everywhere in M implies in general certain conditions that we will investigate for explicit fields and which are the fundamental of what we will call topological quantization.

Let us be more specific. The idea behind the introduction of a differentiable manifold as the underlying geometric structure of spacetime is that in this way one can guarantee the existence of coordinate sets which cover the entire manifold. For the approach we are using here, this is equivalent to having a finite number of sets of orthonormal frames covering the manifold. Let $\{U_\alpha\}$ be an open covering of M, i.e. $\bigcup_\alpha U_\alpha = M$. By definition, a differential manifold of dimension n is equipped with an atlas (U_α, ϕ_α), where ϕ_α is a homeomorphism from U_α onto an open subset of the Euclidean space $\mathbf{R}^n$. If we consider two arbitrary open subsets U_i, $U_j \in \{U_\alpha\}$ such that $U_i \bigcap U_j \neq \emptyset$, then in the intersection region the map $\phi_i \circ \phi_j^{-1}$ is a C^∞ homeomorphism of (open subsets of) $\mathbf{R}^n$. Let $\tilde{\phi}_i$ be the map from U_i into the vector space of 1-forms $\Lambda^1(U_i, so(1,3))$ that allows us to introduce the orthonormal frame e_i in U_i. Notice that the index i labels different sets of orthonormal frames and does not refer to any specific component of the frame. The orthonormal frame e_j attached to U_j by means of $\tilde{\phi}_j$, is related to e_i through an $SO(1,3)$ matrix, $e_i = \Lambda_{ij}e_j$. It is clear that in the intersection region the compatibility condition $\tilde{\phi}_i \circ \tilde{\phi}_j^{-1} = \Lambda_{ij}$ must be satisfied. On U_i we can also introduce a spin connection 1-form ω_i and a curvature 2-form Ω_i, according to Eqs.(1) and (2), respectively. These are related to the connection and curvature in U_j by means of (no summation over repeated indices)

$$\omega_i = \Lambda_{ij}\omega_j\Lambda_{ij}^{-1} + \Lambda_{ij}d\Lambda_{ij}^{-1} \ , \qquad \Omega_i = \Lambda_{ij}\Omega_j\Lambda_{ij}^{-1} \ . \tag{6}$$

Consider now a third open subset $U_k \in \{U_\alpha\}$ such that $U_i \bigcap U_j \bigcap U_k \neq \emptyset$. Accordingly, in

the intersection region we have that $\tilde{\phi}_i \circ \tilde{\phi}_k^{-1} = \Lambda_{ik}$ and $\tilde{\phi}_j \circ \tilde{\phi}_k^{-1} = \Lambda_{jk}$. Then, it follows that

$$\Lambda_{ij}\Lambda_{jk} = \Lambda_{ik} \ . \tag{7}$$

This allows us to formulate the following:

Theorem 1: A solution of Einstein's vacuum field equations can be represented by a unique 10-dimensional principal fiber bundle P with the spacetime M as the base space, the Lorentz group as the structure group (isomorphic to the standard fiber) and a connection with values in the Lie algebra of the Lorentz group.

Proof: A standard theorem (the reconstruction theorem) in differential geometry[5,6] states that a fiber bundle is uniquely specified by the base space, the standard fiber, a structure group which is effectively represented on the fiber and a family of transition functions, with values in the structure group, satisfying the cocycle condition. In the case of a principal fiber bundle P, the fiber is isomorphic to the structure group which is naturally represented on itself by left translations. For a solution of the vacuum field equations, given by an orthonormal frame e, we can take the spacetime manifold M described above as the base space. The structure group is identified as $SO(1,3)$. The transition functions are given by the elements $\Lambda_{ij} : U_i \cap U_j \to SO(1,3)$ and satisfy the cocycle condition which is given above in Eq.(7). Finally, one can show that it is possible to construct the projection $\pi : P \to M$[6], once the transition functions are given. This shows that all the elements of a principle fiber bundle exist and they can be "glued" together to form the desired bundle by means of the transition functions and the projection π.

Finally, we have to show that there exists a connection $\boldsymbol{\omega}$ in P. By construction, we do have a connection 1-form ω on M, which is the connection associated with the vacuum solution. We will see that it determines a unique connection in P. To this end, we use the following theorem[4] valid for principal fiber bundles.

Theorem 2: Given an open covering $\{U_\alpha\}$ of M, a structure Lie group G with Lie algebra $\mathbf{g}$, a family of local $\mathbf{g}$-valued 1-forms $\omega_i \in \Lambda^1(U_i, \mathbf{g})$ which fulfill the compatibility condition

$$\omega_i = g_{ji}^{-1}\omega_j g_{ji} + g_{ji}^{-1}dg_{ji} \ , \tag{8}$$

where $g_{ji} : U_i \cap U_j \to G$ are elements of G, and a set of local sections $\sigma_i : U_i \to \pi^{-1}(U_i)$ satisfying $\sigma_j = \sigma_i g_{ij}$ on $U_i \cap U_j$, then there is a unique connection $\boldsymbol{\omega}$ on P such that $\omega_i = \sigma_i^*\boldsymbol{\omega}$, where σ_i^* is the pull-back induced by σ_i.

In the case we are considering, the structure group is $SO(1,3)$, the family of 1-forms ω_i is determined by the connection ω defined on each $U_i \in \{U_\alpha\}$. The compatibility condition (8) coincides with the transformation property (6), once g_{ji}^{-1} is identified with Λ_{ij}. It remains to show the existence of local sections. Since any fiber bundle accepts a local trivialization which can be defined as $\Psi_i : \pi^{-1}(U_i) \to U_i \times G$, we can introduce a local canonical section on U_i by transferring back to $\pi^{-1}(U_i)$ the section of $U_i \times G$, i.e. by defining $\sigma_i : U_i \to \pi^{-1}(U_i)$ by $\sigma(x) = \Psi_i^{-1}(x, e)$, where $x \in U_i$ and $e = g_{ii}(x)$ is the identity element of G. It is then possible to show[6] that this canonical section satisfies $\sigma_j = \sigma_i g_{ij}$ as required. Thus, according to Theorem 2, there exists a unique connection $\boldsymbol{\omega}$ on P. This ends the proof of Theorem 1.

We have shown that a vacuum solution can naturally be represented as a principal fiber bundle. In all steps of the proof, the local connection ω_i plays an important role and we have assumed that it satisfies the continuity and differentiability conditions on U_i. The question arises whether this assumption can be realized in concrete examples of gravitational configurations. This is exactly the question that we want to address in this work by constructing explicitly the elements of the principal fiber bundles that correspond to given solutions of Einstein's equations. This is what we call the method of "topological quantization". This concept have been used before in the context of diverse monopole configurations[7]. We will see that in the process of constructing a suitable covering $\{U_\alpha\}$ of M, certain "quantization" conditions appear that imply restrictions on the parameters entering the components of the connection.

If it turns out that the constructed principal fiber bundle admits a global section, the bundle is globally trivial and a single connection can be defined everywhere on P. This could happen, for instance, when the base space (spacetime manifold) is contractible. We will see that even in this simple case non trivial conditions arise from the requirement that the connection is regular on all points of the base space. More general cases can be obtained from non contractible manifolds which are very common in general relativity. Explicit solutions of Einstein's equations are usually characterized by the existence of singularities, i.e. regions that can not be described within the formalism of general relativity. In order to properly describe the spacetime manifold we need to "remove" those singular regions from the manifold. This procedure can be used to obtain non contractible base spaces for which we can expect that non trivial conditions arise from the application of the method of topological quantization. In fact, non contractible base spaces can give rise to globally non

trivial principal fiber bundles. In this case one needs more than one open subset to cover the base space and, consequently, transition functions appear that turn out to generate non trivial "quantization" conditions.

Thus, topological quantization is closely related to the problem of determining whether a principal fiber bundle is globally trivial or not. This is a task that involves the relation between the global topological structure of the base space and the fiber, an issue that is used to perform the classification of bundles and is related to the theory of characteristic classes and topological invariants. Consequently, the method of topological quantization is closely related to the study of the topological structure of the underlying bundle.

From the discussion above it follows that one has (at least) two ways to perform the topological quantization of a given solution of Einstein's equations. The first one consists in using Theorem 1 to construct the corresponding principal fiber bundle P and the connection $\boldsymbol{\omega}$. Then, one can analyze the topological invariants of the bundle. The second method consists in constructing explicitly the covering $\{U_\alpha\}$ of M and the family of connection 1-forms $\{\omega_\alpha\}$ on M for the given solution and extracting from there the quantization conditions. Obviously, both methods must yield the same results. When analyzing explicit examples, however, it is not always easy to construct the unknown principal fiber bundle, whereas the second method is straightforward because we know the connection ω explicitly. For this reason, in this work we will apply mainly the second approach for explicit calculations.

All the above discussion involves only the $so(1,3)-$connection associated to the gravitational action. If an additional matter action is considered, the results can be formulated in the following form.

Theorem 3: A solution of Einstein's field equations coupled to a matter gauge field can be represented by a unique principal fiber bundle with the spacetime M as the base space, the gauge group as the structure group (isomorphic to the standard fiber) and a connection with values in the Lie algebra of the gauge group.

The proof of this Theorem is similar to that of Theorem 1. Indeed, for each open subset $U_i \in \{U_\alpha\}$ we can calculate the corresponding gauge connection A_i with values in the Lie algebra of G. The proof can then be carried out in a similar manner with ω_i replaced by A_i, Λ_{ij} replaced by $\gamma_{ij} \in G$, and $g_{ji}^{-1} = \gamma_{ij}$. Consequently, in the case of additional matter gauge fields we can construct on M a principal fiber bundle for each additional gauge connection. So we are lead to the concept of "multiple" principal fiber bundles that can be constructed

on the same base space M. Since the method of topological quantization can be applied to each bundle separately, one could expect different sets of quantization conditions from each bundle. The compatibility of these sets is an issue that can be treated at the level of explicit gravitational configurations.

In the following sections we will apply Theorems 1 and 3 to different gravitational configurations.

III. CYLINDRICALLY SYMMETRIC GRAVITATIONAL FIELDS

Cylindrically symmetric vacuum gravitational configurations can be described by means of the Einstein-Rosen line element which in an orthormal frame can be written as[9]

$$e^0 = \exp(\gamma - \psi)dt \ , \quad e^1 = \exp(\gamma - \psi)d\rho \ , e^2 = \exp(\psi)dz \ , \quad e^3 = \rho \exp(-\psi)d\varphi \ , \qquad (9)$$

where t, ρ, z, and φ are cylindrical coordinates and the functions ψ and γ depend on t and ρ only. For the sake of simplicity, here we restrict ourselves to the case in which the Killing vector fields ∂_z and ∂_φ are hypersurface orthogonal. The corresponding vacuum field equations can be reduced to a second-order differential equation for the function ψ

$$\psi'' + \frac{1}{\rho}\psi' - \ddot{\psi} = 0 \ , \qquad (10)$$

and two first-order differential equations for γ

$$\gamma' = \rho(\dot{\psi}^2 + \psi'^2) \ , \quad \dot{\gamma} = 2\rho\dot{\psi}\psi' \ , \qquad (11)$$

where $\dot{\psi} = \partial\psi/\partial t$, $\psi' = \partial\psi/\partial\rho$, etc. The orthonormal frame (9) is defined up to an arbitrary transformation of the Lie group $SO(1,3)$. If we envision the spacetime as the 4-dimensional base manifold and attach a copy of $SO(1,3)$ at each point of the base manifold, we obtain the 10-dimensional principal fiber bundle considered in Theorem 1. Using the first structure equation (1) it is straightforward to calculate the components of the connection 1-form $\omega^a{}_b$ which can be decomposed as $\omega^a{}_b = \omega^a{}_{b\,\mu}dx^\mu$. It is in this decomposition that the endomorphic character of the connection (an 1-form with values in the Lie algebra $so(1,3)$) becomes plausible[8]. From Eq.(9) we obtain

$$
\begin{aligned}
\omega^0{}_{1\,t} &= \gamma' - \psi' \ , \quad \omega^0{}_{1\,\rho} = \dot{\gamma} - \dot{\psi} \ , \\
\omega^0{}_{2\,z} &= \dot{\psi}\exp(2\psi - \gamma) \ , \quad \omega^1{}_{2\,z} = -\psi'\exp(2\psi - \gamma) \ , \\
\omega^0{}_{3\,\varphi} &= -\rho\dot{\psi}\exp(-\gamma) \ , \quad \omega^1{}_{3\,\varphi} = -(1 - \rho\psi')\exp(-\gamma) \ .
\end{aligned}
\qquad (12)
$$

As described in the last section, we have to demand the regularity of this connection as a condition for constructing the corresponding principal fiber bundle. It is well known[9] that solutions to the field equations can be generated which are everywhere regular with the symmetry axis ($\rho = 0$) as the only possible hypersurface where curvature singularities may appear. Let us suppose that the symmetry axis is free of curvature singularities. Then, there must exist an atlas where the connection is also regular. The field equations (11) implies that at the axis $\dot{\gamma}(\rho \to 0) = \dot{\gamma}_0 = 0$ and $\gamma'(\rho \to 0) = \gamma'_0 = 0$, if $\dot{\psi}(\rho \to 0) = \dot{\psi}_0$ and $\psi'(\rho \to 0) = \psi'_0$ do not diverge. On the other hand, Eq.(10) implies that near the axis $\psi' \propto \rho^\alpha$ with $\alpha > 1$, i.e., $\psi'_0 = 0$, and, consequently, $\ddot{\psi}(\rho \to 0) \propto \rho^{\alpha-1}$. Then, $\dot{\psi}_0$ is at most a constant that can be set equal to zero by means of a coordinate transformation. Thus we have that the regularity condition at the axis implies that $\dot{\gamma}_0 = \gamma'_0 = \dot{\psi}_0 = \psi'_0 = 0$. The same result can be obtained by analyzing the behavior of the curvature Kretschman scalar near the axis. Hence, from Eq.(13) it follows that at the axis $\omega^a{}_b|_{\rho=0} = \omega^a{}_{b\varphi}|_{\rho=0} d\varphi$ with

$$\omega^a{}_{b\varphi}|_{\rho=0} = \exp(-\gamma_0)T_\varphi , \tag{13}$$

where T_φ is one of the generators of the Lie algebra $so(1,3)$:

$$T_\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{14}$$

If we demand that $\exp(-\gamma_0)$ does not diverge, the components $\omega^a{}_{b\varphi}|_{\rho=0}$ are regular, but we still have a singularity in the 1-form $d\varphi = (exp(\psi)/\rho)e^0$. Therefore, the only possibility to get rid of this singularity is to find a gauge transformation such that the new components $\omega'^a{}_{b\varphi}|_{\rho=0}$ vanish identically on the axis. To this end, let us consider the $SO(1,3)$-transformation

$$\Lambda = \exp(\tilde{\varphi}T_\varphi) , \qquad \tilde{\varphi} = \exp(-\gamma_0)\varphi . \tag{15}$$

From Eq.(4) we have that the gauge-transformed 1-form connection is given by

$$\omega' = \exp(\tilde{\varphi}T_\varphi) \, \omega \, \exp(-\tilde{\varphi}T_\varphi) - \exp(-\gamma_0)T_\varphi \, d\varphi , \tag{16}$$

where

$$\exp(\pm\tilde{\varphi}T_\varphi) = 1_{4\times4} \pm \sin\tilde{\varphi} \, T_\varphi + (1 - \cos\tilde{\varphi})T_\varphi^2 , \tag{17}$$

where $1_{4\times 4}$ is the 4×4 unit matrix. The explicit calculation of the components can be carried out in a straightforward manner and leads to

$$
\begin{aligned}
\omega'^{0}{}_{1t} &= (\gamma' - \psi')\cos\tilde\varphi \ , \quad \omega'^{0}{}_{3t} = (\gamma' - \psi')\sin\tilde\varphi \ , \\
\omega'^{0}{}_{1\rho} &= (\dot\gamma - \dot\psi)\cos\tilde\varphi \ , \quad \omega'^{0}{}_{3\rho} = (\dot\gamma - \dot\psi)\sin\tilde\varphi \ , \\
\omega'^{0}{}_{2z} &= \dot\psi\exp(2\psi - \gamma) \ , \quad \omega'^{1}{}_{2z} = -\psi'\exp(2\psi - \gamma)\cos\tilde\varphi \ , \\
\omega'^{2}{}_{3z} &= \psi'\exp(2\psi - \gamma)\sin\tilde\varphi \ , \quad \omega'^{0}{}_{1\varphi} = \rho\dot\psi\exp(-\gamma)\sin\tilde\varphi \ , \\
\omega'^{0}{}_{3\varphi} &= -\rho\dot\psi\exp(-\gamma)\cos\tilde\varphi \ , \quad \omega'^{1}{}_{3\varphi} = \exp(-\gamma_0)[1 - (1 - \rho\psi')\exp(\gamma_0 - \gamma)] \ .
\end{aligned}
\tag{18}
$$

From the last expression it can easily be seen that the gauge-transformed connection vanishes identically on the symmetry axis and no new singularities appear. This has been achieved by means of the gauge transformation (15) which is single-valued only if $\exp(-\gamma_0) = n$, where n is an integer. This, in turn, implies that the gauge-transformed connection is single-valued. Consequently, the condition $\exp(-\gamma_0) = n$ needs to be satisfied for the connection to be well defined. This is an interesting result that can be interpreted as a "quantization" of the energy of cylindrically symmetric gravitational fields. Indeed, the concept of C-energy was introduced by Thorne[10] for gravitational fields described by the Einstein-Rosen line element (9) The quantity $E_c = \gamma_0$ has been shown to represent the (normalized) C-energy density per length unit along the symmetry axis at a given time. In terms of the "quantization" derived above this means that $E_c = -\ln n$, i.e. the C-energy is a discrete quantity. The fact that it is a negative quantity is interpreted by Thorne as an indication of its "non classical" origin. A second expression that can be considered as a (normalized) C-energy density has been introduced by Thorne as $E_c = 1 - \exp(-2\gamma_0)$. In this case, the quantization condition leads to $E_c = 1 - n^2$, an expression that again indicates the discrete character of the C-energy.

We have shown that it is possible to define just one single connection 1-form on the entire Einstein-Rosen spacetime. This means that the base manifold of the principal fiber bundle can be covered by a single open set U and, therefore, can be considered as $\mathbf{R}^4$ which is a contractible manifold. This implies that the corresponding 10-dimensional principal fiber bundle is globally trivial. This is so because we have demanded that the gravitational field be regular at all points of spacetime, including the symmetry axis. If, instead, we would allow singularities on the axis, it would be necessary to "remove" the axis from the base manifold. This would open the possibility of obtaining a non trivial bundle, an issue that

would require an analysis different from the one presented in this section.

IV. THE WEAK GRAVITATIONAL FIELD

Consider the following line element in spherical coordinates t, r, θ, φ:

$$ds^2 = (1 - 2\phi)dt^2 - 2\chi dt d\varphi - (1 + 2\phi)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)] \; , \qquad (19)$$

where ϕ and χ are functions of the spatial coordinates. We assume that $\phi << 1$ and $\chi << 1$ and consider the weak field limit of Einstein's equations in vacuum. An orthonormal frame appropriate for the line element (19) can be written as

$$e^0 = (1 - \phi)dt + \chi d\varphi \; , \quad e^1 = (1 + \phi)dr \; , \quad e^2 = (1 + \phi)d\theta \; , \quad e^3 = (1 + \phi)r \sin\theta \; , \quad (20)$$

where we have neglected all the second order perturbations in ψ and χ.

It is well known[11] that the weak field equations in the Lorentz gauge can be expressed as the Maxwell equations $\tilde{A}_{\mu,\nu}{}^{\nu} = 0$ for the Maxwell potential

$$\tilde{A}_\mu = -\frac{1}{4}(4\phi, \chi) \; , \qquad (21)$$

which is invariant with respect to the transformation $\tilde{A} \to \tilde{A}' = \tilde{A} + df(x^\mu)$, where $f(x^\mu)$ is an arbitrary smooth function. This indicates that the weak field approximation can be interpreted as $U(1)$-gauge theory. To see this explicitly, we introduce the $u(1)-$connection 1-form $A = -i\tilde{A}$. Then, the transformation law (5) is identically satisfied for the $U(1)$-gauge transformation $\gamma = \exp(if(x^\mu))$.

From the Maxwell equations for this case we see that the functions ϕ and χ are decoupled. Let us consider the following special solution

$$A = i\left[\phi dt + \frac{g}{2}(1 + \cos\theta)d\varphi\right] \; , \qquad (22)$$

where ϕ satisfies the differential equation $\phi_{,j}{}^{j} = 0$ $(j = 1, 2, 3)$, and g is a constant. This is the connection defined on the base manifold. To investigate the singularities of the connection (22), it is convenient to represent it in the orthonormal frame (20). Neglecting all the second order perturbations we obtain

$$A_1 = i\left[\phi e^0 + \frac{g}{2}\frac{(1 + \cos\theta)}{r \sin\theta}e^3\right] \; , \qquad (23)$$

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where we have introduced the subscript "1" to identify it as the connection on the open subset U_1 that will be determined below. We can see that there is a first singularity at $r = 0$ which, however, is a true curvature singularity as can be seen by analyzing the corresponding curvature. A second singularity is situated at $\theta = 0$ which does not appear at the level of the curvature. We "eliminate" the true curvature curvature singularity by removing the origin $r = 0$ from the spacetime. Hence, the base manifold M^4 becomes $M^4 = \mathbf{R}^4 - \{\text{world line of } 0\}$. The second singularity at $\theta = 0$, which corresponds to the positive sector, z_+, of the axis z, implies that the connection A_1 is regular on $U_1 = M^4 - \{z_+\}$. This apparent singularity can be eliminated by means of the gauge transformation $\gamma = \exp(ig\varphi)$ which leads to the new connection (up to first order in the perturbation)

$$A_2 = i \left[\phi e^0 + \frac{g}{2} \frac{(-1 + \cos\theta)}{r \sin\theta} e^3 \right] . \tag{24}$$

In fact, the singularity at $\theta = 0$ has been removed, but a new singularity has appeared at $\theta = \pi$. Consequently, the connection A_2 is regular only in the open subset $U_2 = M^4 - \{z_-\}$, where $\{z_-\}$ denotes the negative $z-$axis.

The subsets U_1 and U_2 define a covering of the base manifold. In the intersection region $U_1 \cap U_2$ the two connections are related by means of the transition function $g_{12} = \exp(ig\varphi)$ which is single-valued only if $g = n$, where n is an integer. To interpret this result we have to find out the physical significance of the parameter g. This can be done, for instance, by calculating the multipole moments of this solution[12]. To do this, it is necessary to specify the function ϕ and we chose the simple solution $\phi = m/r$, where m is a constant. Then, it can be shown that the parameter m represents the monopole moment of a mass distribution and gm corresponds to the monopole moment of an angular momentum distribution. This implies that g represents the gravitomagnetic monopole per mass unit which, according the the quantization condition obtained above, can take only discrete values.

This example reminds us the case of a magnetic monopole in electrodynamics. Indeed, we have chosen the function χ in the connection (22) as the Maxwell potential for Dirac's magnetic monopole. The rest of the analysis is then carried out in a similar way as in standard electrodynamics, due to the analogy between the field equations for the weak field approximation and the Maxwell equations. The result obtained here by analyzing the behavior of the $u(1)-$connection can be reproduced in terms of the topological invariants. The corresponding principal fiber bundle is a 5-dimensional $U(1)$-bundle for which the Chern

invariants can be calculated. The result is again that the constant g becomes quantized.

V. THE REISSNER-NORDSTROM BLACK HOLE

Let us consider the following orthonormal frame for the Reissner-Nordstrom metric:

$$e^0 = \frac{[(r - r_-)(r - r_+)]^{1/2}}{r} dt , \quad e^1 = \frac{r}{[(r - r_-)(r - r_+)]^{1/2}} dr , \quad e^2 = rd\theta , \quad e^3 = r \sin\theta d\varphi ,$$

(25)

with

$$r_\pm = m \pm \sqrt{m^2 - e^2} ,$$

(26)

where m is the mass, e is the net electric charge of the source and the radial values $r_\pm$ correspond to the horizons of the Reissner-Nordstrom black hole. This is a solution of the Einstein-Maxwell equations with the potential $\tilde{A} = -(e/r)dt$. The corresponding $u(1)-$connection $A = -i\tilde{A}$ behaves under a gauge transformation as in Eq.(5). According to the discussion of Section II and Theorem 3, there exists a principal fiber bundle which can be constructed by attaching at each point of the spacetime the fiber $U(1)$.

In this section we will explore the conditions that have to be satisfied on the base manifold for constructing that bundle. To investigate the critical points of the connection 1-form we represent it in the orthonormal frame (25). Then

$$A = ie[(r - r_-)(r - r_+)]^{-1/2} e^0 .$$

(27)

This connection diverges at $r = r_-$ and $r = r_+$, whereas the corresponding field strength is regular at those hypersurfaces. To remove these singularities, we first apply the gauge transformation $\gamma_1 = \exp(iet/r_-)$ on (27) and obtain

$$A_1 = -i\frac{e}{r_-} \left(\frac{r - r_-}{r - r_+}\right)^{1/2} e^0 ,$$

(28)

a $u(1)-$connection which is regular at $r = r_-$, but diverges at $r = r_+$. On the other hand, we can also apply the transformation $\gamma_2 = \exp(iet/r_+)$ on (27). The resulting connection

$$A_2 = -i\frac{e}{r_+} \left(\frac{r - r_+}{r - r_-}\right)^{1/2} e^0$$

(29)

is regular at $r = r_+$, but diverges at $r = r_-$. Thus, we have obtained two different connections with different divergences. Let us choose the open subsets $U_1 = (0, r_+)$ and $U_2 = (r_-, \infty)$.

This set of open subsets covers the radial coordinate r completely so that the subsets $U_1 \times \mathbf{R}^3$ and $U_2 \times \mathbf{R}^3$ are a covering of the base manifold M^4. Then, the connections A_1 and A_2 are well defined on U_1 and U_2, respectively. In the intersection region $U_1 \cap U_2 = (r_-, r_+)$, the connections A_1 and A_2 have to be related by means of the transition function $g_{12} \in U(1)$ which can easily be calculated as

$$g_{12} = \exp\left[ie\left(\frac{1}{r_-} - \frac{1}{r_+} \right) t \right] . \tag{30}$$

The important point about this transition function is that it depends only on the time coordinate t and is defined only on the region contained between the horizons r_- and r_+. On the other hand, it is well known[13] that in this region the coordinate t is not timelike but *spacelike*. Indeed, one of the interesting aspects of the region (r_-, r_+) is that the coordinates t and r interchange their role: what was the radial direction becomes timelike, and the timelike direction becomes spacelike. Therefore, we are allowed to consider t as an angle coordinate $0 \leq t \leq 2\pi$ inside the horizons. This is a consistent procedure that can be carried out explicitly for all black hole vacuum stationary solutions[14,15] and can easily be generalized to the case of electrovacuum stationary axisymmetric solutions. Therefore, if t is a compact and periodic coordinate inside the horizons, the transition function (30) is single-valued only if the coefficient in front of t is an integer, i.e.

$$e\left(\frac{1}{r_-} - \frac{1}{r_+} \right) = \frac{2}{e}\sqrt{m^2 - e^2} = n . \tag{31}$$

This represents a relationship between the physical parameters which describe the black hole. This specific combination of m and e can take only discrete values. Notice that in the special case of an extreme black hole, $e = m$, the only allowed value is $n = 0$. Moreover, the limiting case $e \to 0$ is not allowed. This is due to the fact that in order to perform this "quantization" we have used the $u(1)$−electromagnetic connection which does not exist in the case $e = 0$.

Since the spacetime possesses a curvature singularity at $r = 0$, we have to remove the world line of this event from the base space. This implies that the corresponding principal fiber bundle is not globally trivial. This can be seen explicitly by calculating the topological invariant, which in this case is given in terms of the Chern-form $c = -(e/r^2)dt \wedge dr$. The Chern number is obtained by integrating the Chern-form inside the horizon. As expected, we get the value of $4\pi n$, where n is an integer related to the parameters of the

Reissner-Nordstrom black as given in (31). This represents an alternative derivation of the quantization condition. To verify that this result is also independent of the coordinates, we have performed a similar analysis of the Reissner-Nordstrom metric in Kruskal-like coordinates. As expected, the quantization condition (31) appears in a similar manner.

To conclude this section, it is worth mentioning that a similar analysis can be performed for the Kerr-Newman black hole. It turns out that in this case it is necessary to introduce again two open subsets in order to cover the entire spacetime manifold. The transition function is defined in the region contained between the horizons, where the coordinate t is spacelike, and the quantization condition can be written as

$$\frac{2e^3\sqrt{m^2 - a^2 - e^2}}{e^4 + 4a^2m^2} = n \ , \tag{32}$$

where a represents the angular moment per unit mass of the black hole. As expected, in the limiting case $a = 0$ we recover the expression (31) for the Reissner-Nordstrom black hole.

VI.   CONCLUSIONS

In this paper we have developed the method of topological quantization for gravitational field configurations. First, we have shown that for any vacuum solution of Einstein's field equations there exists a natural unique principal fiber bundle with an $so(1, 3)$−connection. If the gravitational field is minimally coupled to a gauge matter field, there exists also a principal fiber bundle with a matter connection.

This procedure has been carried out explicitly for the gravitational configurations described by cylindrically symmetric spacetimes, the gravitomagnetic monopole in linearized gravity and electrovacuum black holes. In all the cases we have analyzed, the result of the topological quantization is a relationship that indicates the discretization of the parameters entering the corresponding metrics. We have shown that the quantization conditions arise as the result of demanding a regular behavior of the connection on the base manifold. Quantization conditions can appear in globally trivial and non trivial principal fiber bundles. In the latter case, equivalent results can be obtained from the analysis of the corresponding topological invariants.

In this work, we do not analyze the physical significance of the resulting discretization. In particular, it would be interesting to perform the topological quantization of black

holes with respect to the $so(1,3)$−connection which would complement the result of the $u(1)$−connection analyzed here. Preliminary calculations show that the complete quantization of black holes metrics leads to a discretization of the horizon area. This task is currently under investigation[16].

Moreover, in all the examples analyzed in this work we have restricted ourselves to the investigation of the regularity conditions of the connection on the base manifolds. Nevertheless, Theorems 1 and 3 show that there exists an additional connection on the bundle which reduces to the connection on the base manifold, when projected by means of the pull-back of local trivializations. It would be interesting to construct explicitly the connection on the bundle and investigate its properties.

Finally, we should mention that although the term "topological quantization" could be very suggestive, it is by no means a procedure that pretends to compete with already existing and well-developed procedures like canonical quantization. Nevertheless, it is interesting to see that the mere existence of relatively simple geometric structures in gravitational field configurations leads to a discretization of physical parameters, a property that is usually associated with quantization. A much more detailed and deep investigation is necessary in order to establish if topological quantization could be an alternative method to obtain at least partial "quantum" information from a physical system.

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